

Generalized Nonlinear Yule Models

Petr Lansky^α, Federico Polito^β & Laura Sacerdote^β

^αDepartment of Mathematics and Statistics, Masaryk University, Brno, Czech Republic

^βDipartimento di Matematica *G. Peano*, Università degli Studi di Torino, Torino, Italy

May 24, 2016

Abstract

With the aim of considering models with persistent memory we propose a fractional nonlinear modification of the classical Yule model often studied in the context of macroevolution. Here the model is analyzed and interpreted in the framework of the development of networks such as the World Wide Web. Nonlinearity is introduced by replacing the linear birth process governing the growth of the in-links of each specific webpage with a fractional nonlinear birth process with completely general birth rates.

Among the main results we derive the explicit distribution of the number of in-links of a webpage chosen uniformly at random recognizing the contribution to the asymptotics and the finite time correction. The mean value of the latter distribution is also calculated explicitly in the most general case. Furthermore, in order to show the usefulness of our results, we particularize them in the case of specific birth rates giving rise to a saturating behaviour, a property that is often observed in nature. The further specialization to the non-fractional case allows us to extend the Yule model accounting for a nonlinear growth.

Keywords: Yule model; Nonlinear birth process; Fractional calculus; Long memory; Growth with saturation.

MSC2010: 60G22; 60J80; 05C80

1 Introduction and background

The seminal paper [40] was the original inspiration for many studies appeared from the second half of the last century till now. It contains implicitly the preferential attachment paradigm, a concept that had had an enormous success after the appearance of the Barabási–Albert model [2] describing the growth of the World Wide Web (WWW). Preferential attachment models are undoubtedly one of the most studied and appreciated classes of network growth models. The relevant literature is vast and spread in many different fields, ranging from mathematics to physics, computer science, biology, ecology, and many others. It would seem futile to aspire in giving here a thorough review of the literature and we will limit ourselves to cite papers only directly relevant for our work.

The Yule model is a continuous time linear model originally motivated by the study of macroevolutionary dynamics. Later, it was revisited to describe a variety of phenomena, including the growth of webpages and links in the WWW. A second class of preferential attachment models derives from the Simon model [38], a discrete time model originally proposed to describe the count of words in a text and then used in many different fields [37].

Yule, Simon and Barabási–Albert models share asymptotic degree distributions with tails characterized by a power-law behaviour. This fact has determined a frequent confusion among them. In a recent paper [32] we point out the existing relationships and differences between these three models making use of random graph theory. In particular, we show that the Yule model can be obtained as the continuous-time limit of a sequence of suitably rescaled Simon models. The existence of a well determined relationship between Simon and Yule models increases the interest for the Yule model itself that is mathematically more tractable than its discrete-time analogue.

Beside the preferential attachment assumption, further hypotheses characterize Yule, Simon and Barabási–Albert models. In fact they are all Markovian and share linear growth rates. The Markov property of the Yule model is determined by the intrinsic exponential nature of the waiting times for the appearance of genera and of species for each newly created genus (in the original formulation, see [40]). In the sequel we will refer to webpages in place of genera and to in-links in substitution to species.

The success in the modelling a diversified set of phenomena has increased the interest in these models and has suggested to attempt possible improvements. Generalizations of the Yule model have already appeared in the literature [e.g. 24, 36]. In Lansky et al. [21] we show that the introduction of the detachment of in-links leads to a good fit of recent WWW data. We realized, however, that even a generalized Yule model (without detachment) in the cases of non Markov and nonlinear rates models, or even only nonlinear rates, was not yet studied in the literature, at the best of our knowledge.

In Section 2 we introduce a generalization of the Yule model removing the Markov hypothesis and assuming nonlinear rates for the number of in-links growth. In Section 3.3 we exemplify our model in the special case of rates of in-links characterized by saturation and in presence of non-Markovian memory for the in-link growth. To pursue this last aim we substitute the geometric law (corresponding to a linear birth process) for the growth of the number of in-links by a different class of counting processes called fractional nonlinear birth processes. These processes are characterized by a parameter that accounts for the length of the memory of the process. They coincide with the classical nonlinear birth processes for a specific choice of the characterizing parameter. Notice that we consider processes with general nonlinear birth rates. The only assumption we introduce is that the rates are such that explosion of the processes in finite time is not allowed.

Fractional point processes in the recent years have been object of intense study and development starting from the simplest ones such as the fractional homogeneous Poisson process or the fractional linear birth process and going further till processes defined via fairly complex random time-changes. For the sake of clarity and completeness, in the Appendix A we report a brief construction of the fractional nonlinear birth processes. Here we limit ourselves to refer to some of the papers present in the literature such as Laskin [22], Beghin and Orsingher [4], Mainardi et al. [23], Politi et al. [33], Cahoy and Polito [8], Meerschaert et al. [25], Garra et al. [15], Beghin and D’Ovidio [3], Orsingher and Polito [30]. The fractional nonlinear birth process was introduced in Orsingher and Polito [29] and also studied in Orsingher and Polito [31]. Some specific cases and related models are given in Cahoy and Polito [6] and Garra et al. [16].

2 The Classical Yule Model

We consider the growth of a random network described through a Yule model [32]. An initial webpage appears with a single in-link at time $t = 0$. Then each in-link starts duplicating at a constant rate $\lambda > 0$. Hence, the number of in-links for each webpage evolves as a homogeneous linear birth process (Yule process). In turn, new webpages (each of them with a single in-link) are created at a constant rate $\beta > 0$. Similarly to in-links belonging to the same webpage, also webpages develop independently as a Yule process of parameter β .

Let us indicate with $N_\beta(t)$ the process counting the number of webpages and with $N_\lambda(t)$ that counting the number of in-links of a given webpage. Recall now that the state probability distribution $\mathbb{P}(N_\lambda(t) = n)$, $n \geq 1$, of a Yule process is geometric and reads [1]

$$\mathbb{P}(N_\lambda(t) = n) = e^{-\lambda t} (1 - e^{-\lambda t})^{n-1}, \quad n \geq 1, t \geq 0. \quad (2.1)$$

Conditioning on the number of webpages at time t , the random instants at which new webpages appear are distributed as the order statistics of iid random variables with distribution function [27, 11, 13, 35]

$$\mathbb{P}(T \leq \tau) = \frac{e^{\beta\tau} - 1}{e^{\beta t} - 1}, \quad 0 \leq \tau \leq t. \quad (2.2)$$

Our purpose is to identify the probability distribution of the size of a webpage extracted uniformly at random at time t . In order to do so, denote this random quantity with \mathcal{N}_t^Y and call $\mathcal{N}^Y = \lim_{t \rightarrow \infty} \mathcal{N}_t^Y$. We can write that,

$$\mathbb{P}(\mathcal{N}_t^Y = n) = \frac{\beta}{1 - e^{-\beta t}} \int_0^t e^{-\beta y} e^{-\lambda y} (1 - e^{-\lambda y})^{n-1} dy, \quad n \geq 1, t \geq 0, \quad (2.3)$$

as the age $t - T$ of the randomly selected webpage appeared at time T is distributed as a truncated exponential random variable. In the limit for $t \rightarrow \infty$ we clearly obtain

$$\mathbb{P}(\mathcal{N}^Y = n) = \int_0^\infty \beta e^{-\beta y} e^{-\lambda y} (1 - e^{-\lambda y})^{n-1} dy, \quad n \geq 1, \quad (2.4)$$

which leads after some steps to

$$\mathbb{P}(\mathcal{N}^Y = n) = \frac{\beta}{\lambda} \frac{\Gamma(n) \Gamma\left(1 + \frac{\beta}{\lambda}\right)}{\Gamma\left(n + 1 + \frac{\beta}{\lambda}\right)}, \quad n \geq 1. \quad (2.5)$$

Distribution (2.5) is known as Yule or Yule–Simon distribution [18].

3 Generalized Nonlinear Yule Model

As outlined in the introductory section, our aim is to introduce a generalization of the Yule model presented in the preceding section by allowing non-Markov dependence and full nonlinear rates for the birth process governing the developing of the in-links. Let us thus consider a Yule-like model composed by

- a homogeneous Yule process of rate $\beta > 0$ for the development of webpages;
- independent copies of a fractional nonlinear birth process $\mathfrak{N}^\nu(t)$, $t \geq 0$, of parameter of fractionality $\nu \in (0, 1)$, for the development of in-links for each webpage;
- rates λ_k , $k = 1, 2, \dots$, for the fractional nonlinear birth process such that explosions are not allowed, that is we admit only a finite number of jumps for any finite time. For this it is sufficient to assume that $\sum_{k=1}^\infty \lambda_k^{-1} = \infty$.

The basic properties of the fractional nonlinear birth process are recalled in the Appendix A. For a quick comparison with the classical nonlinear birth process see Table 1.

Let us now call ${}_t\mathfrak{N}^\nu$ the number of in-links of a webpage chosen uniformly at random at time t and define $\mathfrak{N}^\nu = \lim_{t \rightarrow \infty} {}_t\mathfrak{N}^\nu$. For this generalized Yule model we will evaluate the explicit distribution of ${}_t\mathfrak{N}^\nu$ and of \mathfrak{N}^ν in Section 3.1 and their mean value in Section 3.2. Two specific examples with rates allowing saturation are analyzed in Section 3.3.

3.1 Distribution of ${}_t\mathfrak{N}^\nu$

With the convention here and in the rest of the paper that empty products equal unity, we can evaluate the distribution of the random number of in-links ${}_t\mathfrak{N}^\nu$ by conditioning on the random instant of time at which a webpage is created (and therefore the in-links process of that webpage begins).

We make use here of the theoretical results contained in the papers [26, 10, 14, 34]. Conditioning on the number of webpages present at the observation time t , the random instants at which new webpages are created are distributed as the order statistics of independent and identically distributed random variables with distribution function

$$\mathbb{P}(\mathcal{T} \leq y) = \frac{e^{\beta y} - 1}{e^{\beta t} - 1}, \quad y \in [0, t]. \quad (3.1)$$

Hence $\mathcal{Q} = t - \mathcal{T}$, the random evolution time of the conditioned fractional nonlinear birth process $\mathfrak{N}^\nu(t)$, $t \geq 0$, is distributed as a truncated exponential random variable. It immediately follows that the distribution of ${}_t\mathfrak{N}^\nu$ can be determined by randomization with respect to \mathcal{T} :

$$\mathbb{P}({}_t\mathfrak{N}^\nu = n) = \mathbb{E}_{\mathcal{T}}\mathbb{P}(\mathfrak{N}^\nu(t) = n | \mathfrak{N}^\nu(\mathcal{T}) = 1), \quad n \geq 1. \quad (3.2)$$

Working out formula (3.2) we obtain

$$\begin{aligned} \mathbb{P}({}_t\mathfrak{N}^\nu = n) &= \frac{\beta}{1 - e^{-\beta t}} \int_0^t e^{-\beta y} \mathbb{P}(\mathfrak{N}^\nu(y) = n) dy \\ &= \frac{\beta}{1 - e^{-\beta t}} \left[\int_0^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}^\nu(y) = n) dy - \int_t^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}^\nu(y) = n) dy \right] \\ &= \frac{1}{1 - e^{-\beta t}} \left[\beta^\nu \frac{\prod_{r=1}^{n-1} \lambda_r}{\prod_{r=1}^n (\beta^\nu + \lambda_r)} - \beta \int_t^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}^\nu(y) = n) dy \right]. \end{aligned} \quad (3.3)$$

In the last step we used the explicit form of the Laplace transform of the state probabilities of the fractional nonlinear birth process (formula (A.7)). Note that in the last line of (3.3) we have actually separated the limiting state distribution and the correction for a finite time t . Indeed by letting $t \rightarrow \infty$ the limiting distribution reads

$$\mathbb{P}(\mathfrak{N}^\nu = n) = \beta^\nu \frac{\prod_{r=1}^{n-1} \lambda_r}{\prod_{r=1}^n (\beta^\nu + \lambda_r)}, \quad n \geq 1. \quad (3.4)$$

The above formula (3.4) can be reparametrized by setting $\rho_r^{-1} = \beta^\nu / \lambda_r$, obtaining

$$\mathbb{P}(\mathfrak{N}^\nu = n) = \frac{\rho_n^{-1}}{\prod_{r=1}^n (\rho_r^{-1} + 1)}, \quad n \geq 1. \quad (3.5)$$

When the rates are all different we can further work out (3.3) as follows. Let us first evaluate the lower and upper incomplete Laplace transforms of the Mittag-Leffler function E_ν (see the Appendix A, formula (A.5) for the definition of the Mittag-Leffler function),

$$\begin{aligned} L &= \int_0^t e^{-\beta y} E_\nu(-\lambda y^\nu) dy = \int_0^t e^{-\beta y} \sum_{r=0}^\infty \frac{(-\lambda y^\nu)^r}{\Gamma(\nu r + 1)} dy \\ &= \sum_{r=0}^\infty \frac{(-\lambda)^r}{\Gamma(\nu r + 1)} \int_0^t e^{-\beta y} y^{\nu r} dy = \beta^{-1} \sum_{r=0}^\infty \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda}{\beta^\nu} \right)^r, \end{aligned} \quad (3.6)$$

$$\begin{aligned} U &= \int_t^\infty e^{-\beta y} E_\nu(-\lambda y^\nu) dy = \int_t^\infty e^{-\beta y} \sum_{r=0}^\infty \frac{(-\lambda y^\nu)^r}{\Gamma(\nu r + 1)} dy \\ &= \sum_{r=0}^\infty \frac{(-\lambda)^r}{\Gamma(\nu r + 1)} \int_t^\infty e^{-\beta y} y^{\nu r} dy = \beta^{-1} \sum_{r=0}^\infty \frac{\Gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda}{\beta^\nu} \right)^r, \end{aligned} \quad (3.7)$$

where $\gamma(a, b)$ and $\Gamma(a, b)$ are respectively the lower and upper incomplete Gamma functions, and $\beta > 0$. Clearly,

$$\lim_{t \rightarrow 0} U = \lim_{t \rightarrow \infty} L = L + U = \frac{\beta^{\nu-1}}{\beta^\nu + \lambda} = \mathcal{L}(E_\nu(-\lambda t^\nu))(\beta). \quad (3.8)$$

By using the state probabilities distribution of the fractional nonlinear birth process (Appendix A, formula (A.4)) and formula (3.6), the distribution (3.3) (first line) can be written as

$$\mathbb{P}({}_t\mathfrak{N}^\nu = n) = \frac{1}{1 - e^{-\beta t}} \prod_{h=1}^{n-1} \lambda_h \sum_{m=1}^n \frac{1}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)} \sum_{r=0}^\infty \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^\nu} \right)^r, \quad n \geq 1, \quad (3.9)$$

or, equivalently, by using formula (3.7) and the third line of (3.3), that is highlighting the asymptotics, as

$$\mathbb{P}(t\mathfrak{N}^\nu = n) = \frac{1}{1 - e^{-\beta t}} \prod_{h=1}^{n-1} \lambda_h \sum_{m=1}^n \frac{1}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)} \left[\frac{\beta^\nu}{\beta^\nu + \lambda_m} - \sum_{r=0}^{\infty} \frac{\Gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^\nu} \right)^r \right], \quad n \geq 1. \quad (3.10)$$

In the next three remarks we examine three different specific cases of interest. In Remark 3.1 fractionality is suppressed by considering $\nu = 1$. In this case the model is a generalized Yule model in which a nonlinear birth process represents the in-links growth processes. In Remark 3.2 instead we retain fractionality ($\nu \in (0, 1)$) but with linear rates ($\lambda_r = \lambda r$). Remark 3.3 recovers the known form of the in-degree distribution of a webpage chosen uniformly at random in the classical Yule model for any fixed time t as a special case of our more general formula.

Remark 3.1. *In the non fractional case, that is $\nu = 1$, we can further simplify the above probability mass function (3.9) obtaining, after some steps,*

$$\mathbb{P}(t\mathfrak{N}^1 = n) = \frac{\beta}{1 - e^{-\beta t}} \prod_{r=1}^{n-1} \lambda_r \sum_{m=1}^n \frac{1}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)} \left(\frac{1 - e^{-(\beta + \lambda_m)t}}{\beta + \lambda_m} \right), \quad n \geq 1, \quad (3.11)$$

Remark 3.2. *For linear rates, $\lambda_r = \lambda r$, $r \geq 1$, the fractional nonlinear birth process $\mathfrak{N}^\nu(t)$ coincides with the fractional Yule process $\mathfrak{N}_{lin}^\nu(t)$ (see the Appendix A). The distribution (3.3) can be written, for $n \geq 1$, as*

$$\mathbb{P}(t\mathfrak{N}_{lin}^\nu = n) = \frac{\beta}{1 - e^{-\beta t}} \left[\int_0^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}_{lin}^\nu(y) = n) dy - \int_t^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}_{lin}^\nu(y) = n) dy \right]. \quad (3.12)$$

By making use of the simpler form of the state probability distribution of the fractional Yule process (see the Appendix A, formula (A.9)) and the Laplace transform of the Mittag-Leffler function (A.6) we obtain that

$$\begin{aligned} \int_0^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}_{lin}^\nu(y) = n) dy &= \int_0^\infty e^{-\beta y} \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} E_\nu(-\lambda j y^\nu) dy \\ &= \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} \frac{\beta^{\nu-1}}{\beta^\nu + \lambda j} = \beta^{\nu-1} \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} \int_0^\infty e^{-w(\beta^\nu + \lambda j)} dw \\ &= \beta^{\nu-1} \int_0^\infty e^{-w(\beta^\nu + \lambda)} \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j e^{-w\lambda j} = \beta^{\nu-1} \int_0^\infty e^{-w\beta^\nu} e^{-w\lambda} (1 - e^{-w\lambda})^{n-1} dw \\ &= \beta^{\nu-1} \int_0^1 e^{\frac{\beta^\nu}{\lambda} \log y} (1-y)^{n-1} \frac{dy}{\lambda} = \frac{\beta^{\nu-1}}{\lambda} \int_0^1 y^{\frac{\beta^\nu}{\lambda}} (1-y)^{n-1} dy \\ &= \frac{\beta^{\nu-1}}{\lambda} \frac{\Gamma\left(\frac{\beta^\nu}{\lambda} + 1\right) \Gamma(n)}{\Gamma\left(\frac{\beta^\nu}{\lambda} + n + 1\right)}. \end{aligned} \quad (3.13)$$

Therefore we can write the distribution of interest for each finite time t as

$$\begin{aligned} \mathbb{P}(t\mathfrak{N}_{lin}^\nu = n) &= \frac{\beta}{1 - e^{-\beta t}} \left[\frac{\beta^{\nu-1}}{\lambda} \frac{\Gamma\left(\frac{\beta^\nu}{\lambda} + 1\right) \Gamma(n)}{\Gamma\left(\frac{\beta^\nu}{\lambda} + n + 1\right)} - \int_t^\infty e^{-\beta y} \mathbb{P}(\mathfrak{N}_{lin}^\nu(y) = n) dy \right] \end{aligned} \quad (3.14)$$

$$= \frac{1}{1 - e^{-\beta t}} \left[\frac{\beta^\nu \Gamma\left(\frac{\beta^\nu}{\lambda} + 1\right) \Gamma(n)}{\lambda \Gamma\left(\frac{\beta^\nu}{\lambda} + n + 1\right)} - \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} \sum_{r=0}^{\infty} \frac{\Gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda}{\beta^\nu} m\right)^r \right],$$

and the limiting distribution as

$$\mathbb{P}(\mathfrak{N}_{lin}^\nu = n) = \lim_{t \rightarrow \infty} \mathbb{P}(t \mathfrak{N}_{lin}^\nu = n) = \frac{\beta^\nu \Gamma\left(\frac{\beta^\nu}{\lambda} + 1\right) \Gamma(n)}{\lambda \Gamma\left(\frac{\beta^\nu}{\lambda} + n + 1\right)}, \quad n \geq 1. \quad (3.15)$$

The probability mass function (3.15) is the usual Yule or Yule–Simon distribution of parameter $\rho^{-1} = \beta^\nu / \lambda$. We remark that from (3.14) it is clear the contribution to the asymptotics and the finite time correction.

Remark 3.3. If $\nu = 1$, equation (3.14) gives us the probability distribution of $t \mathfrak{N}_{lin}^1$, i.e. that of the classical Yule model for a finite time t . Indeed we have that, for $n \geq 1$,

$$\mathbb{P}(t \mathfrak{N}_{lin}^1 = n) = \frac{1}{1 - e^{-\beta t}} \left[\frac{\beta \Gamma\left(\frac{\beta}{\lambda} + 1\right) \Gamma(n)}{\lambda \Gamma\left(\frac{\beta}{\lambda} + n + 1\right)} - \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} \sum_{r=0}^{\infty} \frac{\Gamma(r + 1, \beta t)}{r!} \left(-\frac{\lambda}{\beta} m\right)^r \right],$$

or, in a more compact form,

$$\begin{aligned} \mathbb{P}(t \mathfrak{N}_{lin}^1 = n) &= \frac{\beta}{1 - e^{-\beta t}} \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} \frac{1 - e^{-t(\beta + \lambda j)}}{\beta + \lambda j} \\ &= \frac{1}{1 - e^{-\beta t}} \frac{\beta}{\lambda} \left[\frac{\Gamma\left(\frac{\beta}{\lambda} + 1\right) \Gamma(n)}{\Gamma\left(\frac{\beta}{\lambda} + n + 1\right)} - \text{Be}\left(e^{-\lambda t}, \frac{\beta}{\lambda} + 1, n\right) \right] \\ &= \frac{1}{1 - e^{-\beta t}} \frac{\beta}{\lambda} \text{Be}\left(1 - e^{-\lambda t}, \frac{\beta}{\lambda} + 1, n\right). \end{aligned} \quad (3.16)$$

where $\text{Be}(z; a, b)$ is the incomplete Beta function.

3.2 Mean of $t \mathfrak{N}^\nu$

In this section we derive the explicit form of $\mathbb{E}_t \mathfrak{N}^\nu$, that is the expected value of the number of in-links of a webpage chosen uniformly at random from those present at time t in the generalized Yule model with which we are concerned. By randomizing on the random creation time of the uniformly chosen webpage we have that

$$\mathbb{E}_t \mathfrak{N}^\nu = \frac{\beta}{1 - e^{-\beta t}} \int_0^t e^{-\beta y} \mathbb{E} \mathfrak{N}^\nu(y) dy, \quad t \geq 0. \quad (3.17)$$

In order to obtain an explicit expression we now need to use an explicit form for the expected number of in-links in the fractional nonlinear birth process \mathfrak{N}^ν . To do so we make use of Theorem 3.2 of Orsingher and Polito [31] where the mean value of the fractional nonlinear birth process in the case of rates all different is derived. We have that

$$\mathbb{E} \mathfrak{N}^\nu(t) = 1 + \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) E_\nu(-\lambda_m t^\nu) \right\}, \quad t \geq 0. \quad (3.18)$$

In the next theorem we calculate, for any fixed time t , the expected value of the number of in-links of a webpage chosen uniformly at random from those present at time t .

Theorem 3.1. *If the rates λ_r , $r \geq 1$, are all different we have, for $t \geq 0$,*

$$\mathbb{E}_t \mathfrak{N}^\nu = 1 + \sum_{k=1}^{\infty} \left[1 - \frac{1}{1 - e^{-\beta t}} \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \sum_{r=0}^{\infty} \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^\nu} \right)^r \right]. \quad (3.19)$$

Proof. It follows naturally by inserting (3.18) into (3.17) as follows.

$$\begin{aligned} \mathbb{E}_t \mathfrak{N}^\nu &= \frac{\beta}{1 - e^{-\beta t}} \int_0^t e^{-\beta y} \left[1 + \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) E_\nu(-\lambda_m y^\nu) \right\} \right] dy \\ &= \frac{\beta}{1 - e^{-\beta t}} \left[\int_0^t e^{-\beta y} dy + \sum_{k=1}^{\infty} \left\{ \int_0^t e^{-\beta y} dy - \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \int_0^t e^{-\beta y} E_\nu(-\lambda_m y^\nu) dy \right\} \right] \\ &= 1 + \sum_{k=1}^{\infty} \left[1 - \frac{1}{1 - e^{-\beta t}} \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \sum_{r=0}^{\infty} \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^\nu} \right)^r \right]. \end{aligned} \quad (3.20)$$

□

Remark 3.4. *When $\nu = 1$, that is in the classical non-fractional case, the mean value simplifies to*

$$\mathbb{E}_t \mathfrak{N}^1 = 1 + \sum_{k=1}^{\infty} \left[1 - \frac{\beta}{1 - e^{-\beta t}} \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \frac{1 - e^{-(\beta + \lambda_m)t}}{\beta + \lambda_m} \right]. \quad (3.21)$$

Remark 3.5. *The expected value of the limiting random variable \mathfrak{N}^ν can be determined directly from (3.19) as*

$$\mathbb{E} \mathfrak{N}^\nu = 1 + \sum_{k=1}^{\infty} \left[1 - \sum_{m=1}^k \left(\prod_{\substack{l=1 \\ l \neq m}}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \frac{\beta^\nu}{\beta^\nu + \lambda_m} \right]. \quad (3.22)$$

It is interesting to notice from (3.19) and (3.22) that the effect of the fractional parameter ν which is heavily present in the mean value (3.19) for any fixed time t , determines a change in the parametrization in the limiting mean value (3.22).

3.3 Examples: models with saturation

The general model depicted above can be made more specific seeking for particular properties. For example an interesting behaviour that could possibly lead to a more realistic scenario is when the number of in-links for each webpage has intrinsically a fixed value to which it saturates. A quite general saturating behaviour can be achieved by truncating the rates at $N - 1$ (so that $\lambda_N = 0$). This can be done by considering the rates as the weights of a discrete finite measure on the finite set $\{1, 2, \dots, N - 1\}$.

A further rather general model admitting a saturating behaviour is the one in which the rates specialize as

$$\lambda_j = \eta \left(\frac{j}{N} \right)^{\omega_1} \left(\frac{N - j}{N} \right)^{\omega_2} = \lambda j^{\omega_1} (N - j)^{\omega_2}, \quad (\omega_1, \omega_2) \in [0, \infty) \times (0, \infty), \eta > 0, \quad (3.23)$$

where $\lambda = \eta/N^{\omega_1+\omega_2}$. These rates clearly do not allow explosion in a finite time. Note that we have explicitly excluded the cases in which $\omega_2 = 0$ as this choice implies unbounded growth. By specializing the rates in (3.4), and considering $1 \leq n \leq N$, we have

$$\begin{aligned}
\mathbb{P}(\mathfrak{N}^\nu = n) &= \frac{\beta^\nu \prod_{r=1}^{n-1} (\lambda r^{\omega_1} (N-r)^{\omega_2})}{\prod_{r=1}^n (\beta^\nu + \lambda r^{\omega_1} (N-r)^{\omega_2})} \\
&= \beta^\nu \lambda^{n-1} (n-1)!^{\omega_1} \frac{(N-1)!^{\omega_2}}{(N-n)!^{\omega_2}} \frac{1}{\beta^{\nu n} \prod_{r=1}^n \left(1 + \frac{\lambda}{\beta^\nu} r^{\omega_1} (N-r)^{\omega_2}\right)} \\
&= \left(\frac{\lambda}{\beta^\nu}\right)^{n-1} \frac{\Gamma^{\omega_1}(n) \Gamma^{\omega_2}(N)}{\Gamma^{\omega_2}(N-n+1)} \frac{1}{\prod_{r=1}^n \left(1 + \frac{\lambda}{\beta^\nu} r^{\omega_1} (N-r)^{\omega_2}\right)} \\
&= \rho^{n-1} \frac{\Gamma^{\omega_1}(n) \Gamma^{\omega_2}(N)}{\Gamma^{\omega_2}(N-n+1)} \frac{1}{\prod_{r=1}^n (1 + \rho r^{\omega_1} (N-r)^{\omega_2})}.
\end{aligned} \tag{3.24}$$

3.3.1 First example

A first example is when the nonlinear rates λ_j particularize to $\lambda_j = \lambda(N-j)$, $\lambda > 0$, $1 \leq j \leq N$, where N is the threshold integer value which cannot be crossed: when the process is in state N the birth rate vanishes. This corresponds to $(\omega_1, \omega_2) = (0, 1)$, $\lambda = \eta/N$. In this case, since the rates are all different we can explicitly calculate the Laplace transform of the state distribution of the fractional nonlinear birth process governing the evolution of the in-links for each webpage as follows (or equivalently we suitably specialize the birth rates in formula (A.7)).

$$\begin{aligned}
\mathbb{L}_n(z) &= \prod_{m=1}^{n-1} \lambda(N-j) \sum_{m=1}^n \frac{z^{\nu-1}}{z^\nu + \lambda(N-m)} \frac{1}{\prod_{l=1, l \neq m}^n (\lambda(N-l) - \lambda(N-m))} \\
&= \sum_{j=1}^{n-1} \frac{(N-1)N \dots (N-n+1)}{(m-1)m \dots (m-m+1)(m-m-1) \dots (m-n)} \frac{z^{\nu-1}}{z^\nu + \lambda(N-m)} \\
&= \sum_{m=1}^n \frac{(N-1)!}{(N-n)!(m-1)!(n-m)!} (-1)^{n-m} \frac{z^{\nu-1}}{z^\nu + \lambda(N-m)} \\
&= \binom{N-1}{N-n} \sum_{m=1}^n \binom{n-1}{m-1} (-1)^{n-m} \frac{z^{\nu-1}}{z^\nu + \lambda(N-m)}, \quad 1 \leq n \leq N.
\end{aligned} \tag{3.25}$$

Let us now denote $\mathfrak{N}_{s_1}^\nu$ the size of a randomly chosen webpage for $t \rightarrow \infty$ for this first model allowing saturation. We immediately have for $1 \leq n \leq N$,

$$\mathbb{P}(\mathfrak{N}_{s_1}^\nu = n) = \binom{N-1}{N-n} \sum_{m=1}^n \binom{n-1}{m-1} (-1)^{n-m} \frac{\beta^\nu}{\beta^\nu + \lambda(N-m)}, \tag{3.26}$$

that is to say, by using the usual parametrization $\rho = \lambda/\beta^\nu$,

$$\mathbb{P}(\mathfrak{N}_{s_1}^\nu = n) = \binom{N-1}{N-n} \sum_{m=1}^n \binom{n-1}{m-1} (-1)^{n-m} \frac{1}{1 + \rho(N-m)}. \tag{3.27}$$

Remark 3.6. The case $(\omega_1, \omega_2) = (1, 0)$ corresponds to the classical Yule model described in the introductory section.

3.3.2 Second example

Here we specialize the birth rates as $\lambda_r = \lambda r(N-r)$, $1 \leq r \leq N$. This corresponds to $(\omega_1, \omega_2) = (1, 1)$, $\lambda = \eta/N^2$. If $\mathfrak{N}_{s_2}^\nu$ is the size of a randomly chosen webpage for $t \rightarrow \infty$ for this second model

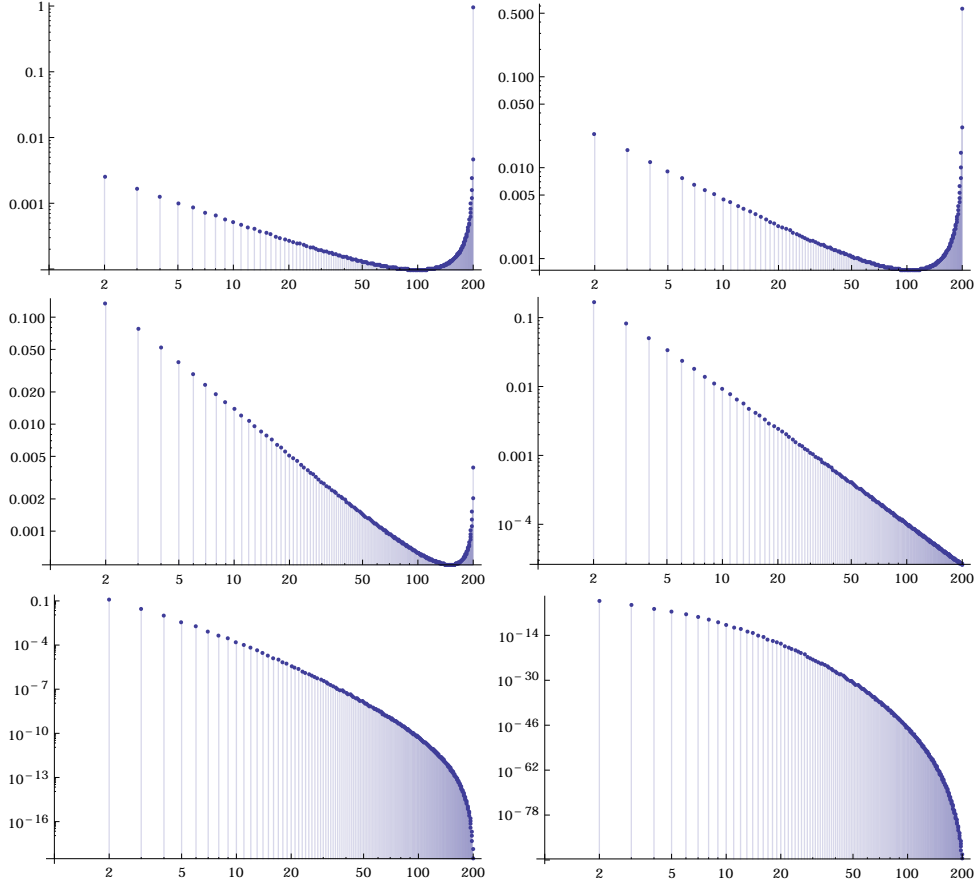


Figure 1: Various plots of $\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n)$. From left to right, top to bottom, the parameters are respectively $(\rho, N) = \{(1, 200), (0.1, 200), (0.01, 200), (0.005, 200), (0.001, 200), (0.0001, 200)\}$.

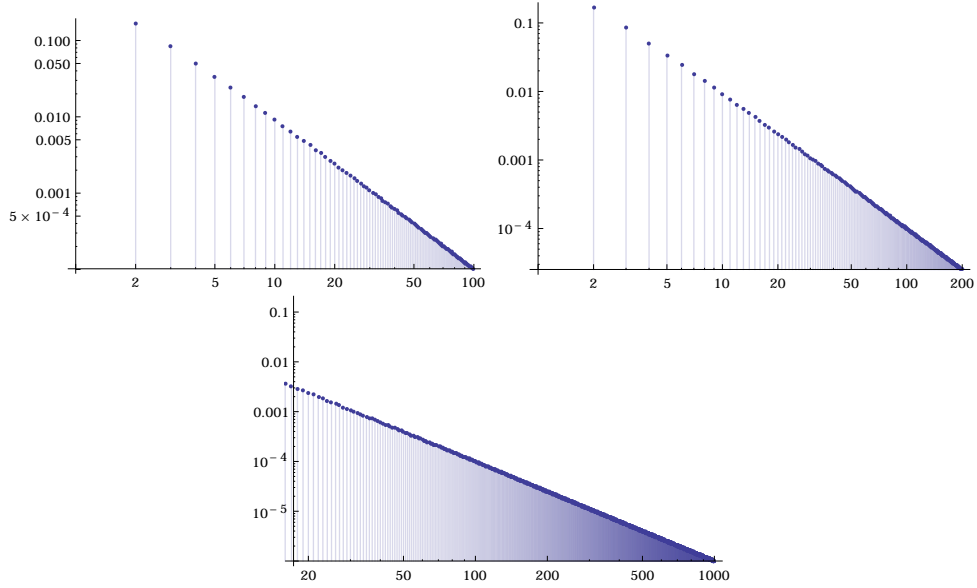


Figure 2: Plots of $\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n)$ when $\rho = 1/(N+1)$. Counterclockwise, starting from the upper-left plot. The parameters are $(\rho, N) = \{(0.01, 100), (0.001, 1000), (0.005, 200)\}$.

allowing saturation, we have

$$\begin{aligned}
\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n) &= \frac{\beta^\nu \prod_{r=1}^{n-1} (\lambda r(N-r))}{\prod_{r=1}^n (\beta^\nu + \lambda r(N-r))} \\
&= \beta^\nu \lambda^{n-1} (n-1)! \frac{(N-1)!}{(N-n)!} \frac{1}{\beta^{\nu n} \prod_{r=1}^n \left(1 + \frac{\lambda}{\beta^\nu} r(N-r)\right)} \\
&= \left(\frac{\lambda}{\beta^\nu}\right)^{n-1} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N-n+1)} \frac{1}{\prod_{r=1}^n \left(1 + \frac{\lambda}{\beta^\nu} r(N-r)\right)}, \quad 1 \leq n \leq N.
\end{aligned} \tag{3.28}$$

Reparametrizing by $\rho = \lambda/\beta^\nu$, we obtain

$$\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n) = \rho^{n-1} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N-n+1)} \frac{1}{\prod_{r=1}^n (1 + \rho r(N-r))}, \quad 1 \leq n \leq N. \tag{3.29}$$

Let us now simplify the product in (3.29) as follows.

$$\begin{aligned}
\prod_{r=1}^n (1 + \rho r(N-r)) &= \prod_{r=1}^n (1 - \rho r^2 + r\rho N) \\
&= (-\rho)^n \prod_{r=1}^n \left(r^2 - 2r\frac{N}{2} - \frac{1}{\rho}\right) \\
&= (-\rho)^n \prod_{r=1}^n \left(r^2 + \frac{N^2}{4} - 2r\frac{N}{2} - \frac{N^2}{4} - \frac{1}{\rho}\right) \\
&= (-\rho)^n \prod_{r=1}^n \left[\left(1 - \frac{N}{2} + r - 1\right)^2 - \frac{1}{4}\left(N^2 + \frac{4}{\rho}\right)\right].
\end{aligned} \tag{3.30}$$

By making the substitutions $a = 1 - N/2$ and $b = (N^2 + 4/\rho)^{1/2}/2$ we have

$$\begin{aligned}
\prod_{r=1}^n (1 + \rho r(N-r)) &= (-\rho)^n \prod_{r=1}^n [(a+r-1)^2 - b^2] \\
&= (-\rho)^n \prod_{r=1}^n (a-b+r-1) \prod_{r=1}^n (a+b+r-1).
\end{aligned} \tag{3.31}$$

Considering that the Pochhammer symbol (e.g. Olver et al. [28], page 136) can be written also as $(c)_n = \prod_{r=1}^n (c+r-1) = \Gamma(c+n)/\Gamma(c)$, we obtain that

$$\prod_{r=1}^n (1 + \rho r(N-r)) = (-\rho)^n \left(1 - \frac{N}{2} - \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)_n \left(1 - \frac{N}{2} + \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)_n. \tag{3.32}$$

Concluding, for $1 \leq n \leq N$, the explicit expression of the probability mass function for the random number of in-links in a webpage chosen uniformly at random can be written as

$$\begin{aligned}
\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n) &= \frac{(-1)^n}{\rho} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N-n+1)} \frac{\Gamma\left(1 - \frac{N}{2} - \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)}{\Gamma\left(n+1 - \frac{N}{2} - \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)} \frac{\Gamma\left(1 - \frac{N}{2} + \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)}{\Gamma\left(n+1 - \frac{N}{2} + \frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}}\right)} \\
&= \rho^{-1} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N-n+1)} \frac{\Gamma\left(\frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}} + 1 - \frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}} + 1 - \frac{N}{2} + n\right)} \frac{\Gamma\left(\frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}} + \frac{N}{2} - n\right)}{\Gamma\left(\frac{1}{2}\sqrt{N^2 + \frac{4}{\rho}} + \frac{N}{2}\right)}.
\end{aligned} \tag{3.33}$$

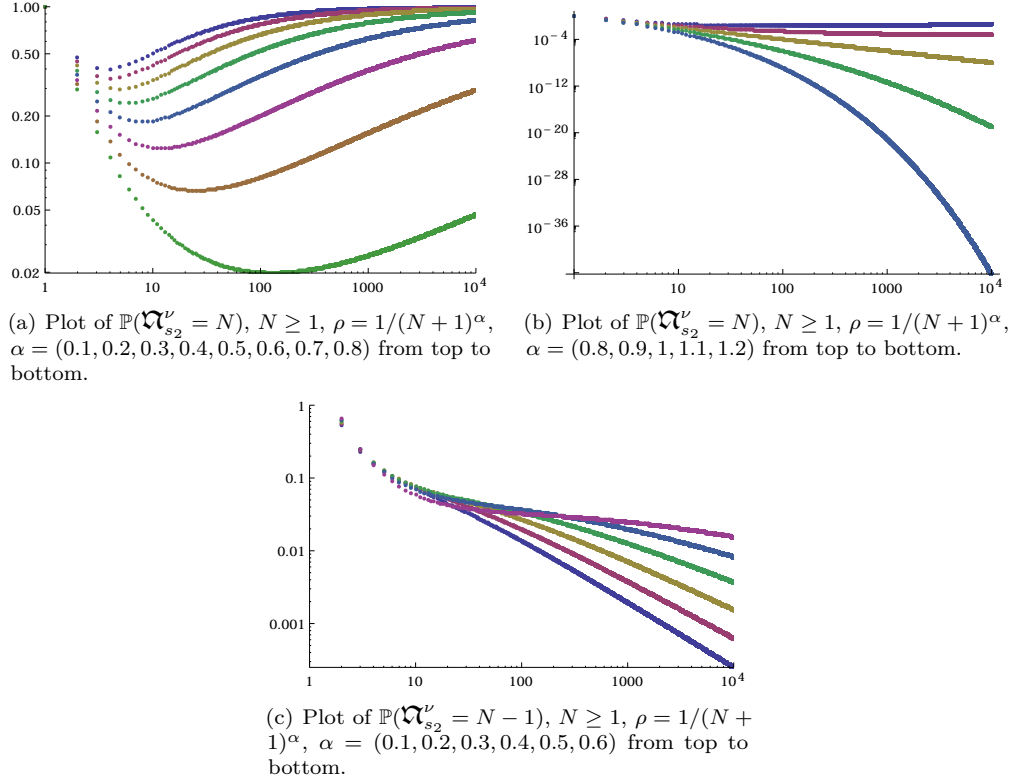


Figure 3: The limit probability of selecting a saturated webpage (figures (a) and (b)) or an almost saturated webpage (figure (c)) with parameter $\rho = 1/(N+1)^\alpha$ for different values of α . Notice how, if $\alpha \in (0, 1)$, the probability tends to unity for $N \rightarrow \infty$ (figures (a) and (b)) while for $\alpha \geq 1$ it decreases towards zero. Accordingly, the probability of selecting a webpage with $N-1$ in-links vanishes asymptotically for $\alpha \in (0, 1)$ (figure (c)).

See in Figure 1 various plots of the probability mass function $\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n)$, $n \geq 1$ (equation (3.33)) for different values of the characterizing parameters, N (threshold at which saturation occurs), and ρ (which takes into account the webpage appearing rate β , the in-links appearing rate λ , and the parameter of fractionality ν). Notably, when $\rho = 1/(N+1)$ the distribution (3.33) simplifies to

$$\mathbb{P}(\mathfrak{N}_{s_2}^\nu = n) = \left(1 + \frac{1}{N}\right) \frac{1}{n^2 + n}, \quad 1 \leq n \leq N. \quad (3.34)$$

The above distribution is shown in Figure 2 for different values of the parameter ρ .

Figure 3 shows a graphical investigation of the asymptotic probability of selecting a saturated ($\mathbb{P}(\mathfrak{N}_{s_2}^\nu = N)$) or almost saturated ($\mathbb{P}(\mathfrak{N}_{s_2}^\nu = N-1)$) webpage with $\rho = 1/(N+1)^\alpha$, $\alpha > 0$, with respect to the threshold N . First note that for $\alpha = 1$ (the case considered in Figure 2) we have a perfect power-law behaviour as formula (3.34) becomes $\mathbb{P}(\mathfrak{N}_{s_2}^\nu = N) = N^{-2}$, $N \geq 1$ (see Figure 3(b)). Interestingly enough, the shape and the limiting value of the probability of selecting a saturated webpage strongly depend on whether α is larger or smaller than unity.

4 Summary and Conclusions

In the paper we have developed a model which generalizes the classical Yule model still maintaining a mathematical tractability. The presented model is interesting in that it admits a nonlinear growth for the number of in-links. More precisely, the generalized model is constructed by considering a fractional nonlinear birth process with completely general rates governing the process of creation

of in-links for each webpage present at a specific time. The distribution of the number of in-links for a webpage chosen uniformly at random is rather different from that of the classical Yule model for each finite time t even considering linear rates. When t goes to infinity (and for linear rates) the obtained distribution is again a Yule distribution but with a different characterizing parameter ρ . This is a particularly important fact in the sense that considering only the limiting behaviour, any empirical Yule distribution recorded on real data can be consequence either of an underlying classical Yule model or of a fractional linear Yule model with the same value of ρ . Notice however that ρ must be interpreted appropriately taking into consideration also the presence of fractionality (given by the value of the parameter ν). The distribution (3.5) is expressed in a very general form due to the fact that the rates are practically unspecified. In Section 3.3, as an example of the different specific cases that can be obtained by specializing the rates, we have chosen rates that produce a saturating behaviour. Also in this rather realistic specific case, the explicit form of the limiting distribution of the number of in-links for a webpage chosen uniformly at random is derived. Figures 1 and 2 show that even specializing the rates, the overall shape of the distribution can be quite different.

A Appendix

For the sake of self-containedness we give here a brief description of the fractional nonlinear birth process. For a quick comparison with the classical nonlinear birth process see Table 1.

Let us consider a population of individuals developing with continuous time t initiated by one single initial progenitor at time $t = 0$. We indicate the random number of individuals in the population for any fixed time t with the random variable $\mathfrak{N}^\nu(t)$. It is known [29] that the state probabilities $p_n^\nu(t) = \mathbb{P}(\mathfrak{N}^\nu(t) = n)$, $n \geq 1$, satisfy the system of difference-differential equations

$$\frac{d^\nu p_n^\nu}{dt^\nu} = -\lambda_n p_n^\nu + \lambda_{n-1} p_{n-1}^\nu, \quad n \geq 1, \quad (\text{A.1})$$

where $p_0^\nu(s) = 0$. Moreover, $p_n^\nu(0) = \delta_{n,1}$, that is the process starts with only one initial progenitor, and the fractional derivative is the Caputo derivative (see e.g. Kilbas et al. [19], Diethelm [12]). Briefly, the Caputo derivative is an integral operator of convolution-type with a singular power-law kernel. The Caputo derivative can be defined in several equivalent ways. We consider here the following form:

Definition A.1 (Caputo derivative). *Let $\alpha > 0$, $m = \lceil \alpha \rceil$, and $f \in AC^m[a, b]$. The Caputo derivative of order $\alpha > 0$ is defined as*

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t (t - s)^{m-1-\alpha} \frac{d^m}{ds^m} f(s) ds. \quad (\text{A.2})$$

In our case we have $\alpha = \nu \in (0, 1)$, $m = 1$, $a = 0$, obtaining

$$\frac{d^\nu}{dt^\nu} p_n^\nu(t) = \frac{1}{\Gamma(1 - \nu)} \int_0^t \frac{d}{ds} p_n^\nu(s) \frac{ds}{(t - s)^\nu}, \quad 0 < \nu < 1. \quad (\text{A.3})$$

It is evident from the above Definition A.1 that the Caputo derivative is a non-local operator in the sense that the integration over the interval $(0, t)$ furnishes the system with a persistent memory. Roughly speaking the first order derivative $\frac{d}{ds} p_n^\nu(s)$ is evaluated along the whole time interval $(0, t)$ and weighted by means of the power-law kernel.

The state probabilities $p_n^\nu(t)$ of the fractional birth process can be explicitly determined and (with the convention that empty products equal unity) have the form (for the nonlinear rates λ_j , $j \geq 1$, all different) [29]

$$p_n^\nu(t) = \mathbb{P}(\mathfrak{N}^\nu(t) = n) = \prod_{j=1}^{n-1} \lambda_j \sum_{m=1}^n \frac{E_\nu(-\lambda_m t^\nu)}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)}, \quad n \geq 1, t \geq 0, \quad (\text{A.4})$$

where $E_\nu(\zeta)$ is the so-called Mittag–Leffler function, a special function defined as

$$E_\nu(\zeta) = \sum_{h=0}^{\infty} \frac{\zeta^h}{\Gamma(\nu h + 1)}, \quad \zeta \in \mathbb{R}, \nu > 0, \quad (\text{A.5})$$

and having Laplace transform

$$\mathcal{L}(E_\nu(-\lambda t^\nu))(z) = \int_0^\infty e^{-zt} E_\nu(-\xi t^\nu) dt = \frac{z^{\nu-1}}{z^\nu + \xi}, \quad \nu > 0, \xi \in \mathbb{R}. \quad (\text{A.6})$$

The state probabilities (A.4) can be actually derived by means of an iterated application of the Laplace transform on the equations (A.1) starting from $n = 1$. For details on this point see Orsingher and Polito [29], Section 2. The Mittag–Leffler function (A.5) is in practice a generalization of the exponential function in the sense that $E_1(\zeta) = \exp(\zeta)$. General properties of the Mittag–Leffler functions are contained in many classical reference books and articles (see e.g. the very recent monograph Gorenflo et al. [17] and the references listed therein).

In the present paper we will often make use of the Laplace transform of the state probabilities (A.4) of the fractional nonlinear birth process. From Orsingher and Polito [29] and Orsingher and Polito [31] we can easily check that that

$$\mathbb{L}_n(z) = \int_0^\infty e^{-zt} p_n^\nu(t) dt = z^{\nu-1} \frac{\prod_{r=1}^{n-1} \lambda_r}{\prod_{r=1}^n (z^\nu + \lambda_r)}, \quad n \geq 1. \quad (\text{A.7})$$

If the rates are all different, equation (A.7) can be written as

$$\mathbb{L}_n(z) = \prod_{r=1}^{n-1} \lambda_r \sum_{m=1}^n \frac{1}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)} \frac{z^{\nu-1}}{z^\nu + \lambda_m}, \quad n \geq 1, \quad (\text{A.8})$$

that is a more manageable form for the purpose of specializing the rates.

For more insights on the properties of the fractional nonlinear birth process see Orsingher and Polito [29, 31]. Here we conclude this section by recalling an interesting representation of the fractional nonlinear birth process as a time-changed process and by giving some details of the specific case in which the rates are linear. Regarding the first point the fractional nonlinear birth process can be constructed as a classical birth process stopped at an independent random time given by the inverse process to an independent ν -stable subordinator. Notice that stable subordinators are increasing spectrally positive Lévy processes with Lévy measure given by $m(dx) = [\nu/\Gamma(1-\nu)] x^{-1-\nu} dx$. For more details on this last point see Bertoin [5], Kyprianou [20].

An interesting particular case is when the rates are linear, i.e. $\lambda_r = \lambda r$. Here the state probability distribution (A.4) specializes in the rather simple form

$$p_n^\nu(t) = \sum_{j=1}^n \binom{n-1}{j-1} (-1)^{j-1} E_\nu(-\lambda j t^\nu), \quad t \geq 0, n \geq 1. \quad (\text{A.9})$$

The geometric distribution of the linear birth process (also known as Yule or Yule–Furry process and indicated in the paper with $\mathfrak{N}_{\text{lin}}^\nu(t)$, $t \geq 0$) is retrieved from (A.9) if the parameter ν is taken equal to unity. Properties of the fractional Yule process have been studied in Uchaikin et al. [39], Orsingher and Polito [29] while estimators for the intensity λ and the fractional parameter ν have been derived in Cahoy and Polito [7, 9].

Acknowledgments

P. Lansky was supported by the Czech Science Foundation project 15-06991S. L. Sacerdote and F. Polito were supported by the projects “Application driven Markov and non Markov models” and “Stochastic modelling beyond diffusions” (Università degli Studi di Torino).

	Nonlinear Birth Process	Fractional Nonlinear Birth Process	Related Formulas
Governing Equations	$\frac{dp_n}{dt^\nu} = -\lambda_n p_n + \lambda_{n-1} p_{n-1}, \quad n \geq 1$	$\frac{d^\nu p_n^\nu}{dt^\nu} = -\lambda_n p_n^\nu + \lambda_{n-1} p_{n-1}^\nu, \quad n \geq 1$	(A.1) (1.3) of [29]
State Probabilities (rates different)	$\prod_{j=1}^{n-1} \lambda_j \sum_{m=1}^n \frac{\exp(-\lambda_m t)}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)}, \quad n \geq 1$	$\prod_{j=1}^{n-1} \lambda_j \sum_{m=1}^n \frac{E_\nu(-\lambda_m t^\nu)}{\prod_{l=1, l \neq m}^n (\lambda_l - \lambda_m)}, \quad n \geq 1$	(A.4) (1.5), (1.11) of [29]
Laplace Transf. of State Probab. (rates different)	$\prod_{r=1}^{n-1} \lambda_r \sum_{m=1}^n \prod_{l=1, l \neq m}^n \frac{1}{(\lambda_l - \lambda_m)} \frac{1}{z + \lambda_m}, \quad n \geq 1$	$\prod_{r=1}^{n-1} \lambda_r \sum_{m=1}^n \prod_{l=1, l \neq m}^n \frac{1}{(\lambda_l - \lambda_m)} \frac{z^{\nu-1}}{z^\nu + \lambda_m}, \quad n \geq 1$	(A.8) (2.26 of [29]
Mean	$1 + \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^k \left(\prod_{l=1, l \neq m}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \exp(-\lambda_m t) \right\}$	$1 + \sum_{k=1}^{\infty} \left\{ 1 - \sum_{m=1}^k \left(\prod_{l=1, l \neq m}^k \frac{\lambda_l}{\lambda_l - \lambda_m} \right) E_\nu(-\lambda_m t^\nu) \right\}$	(3.18) (3.14) of [31]
Density of k th Waiting Time	$\lambda_k \exp(-\lambda_k s), \quad k \geq 1$	$\lambda_k s^{\nu-1} E_{\nu, \nu}(-\lambda_k s^\nu), \quad k \geq 1$	(3.8) of [31]

Table 1: Summary of the basic properties of the nonlinear birth process \mathfrak{N}^ν (second column) compared with the corresponding of the fractional nonlinear birth process \mathfrak{N}^ν (third column). The last column gives references for the corresponding property.

References

- [1] N. T. J. Bailey. *The elements of stochastic processes with applications to the natural sciences*. John Wiley & Sons, Inc., New York-London-Sydney, 1964.
- [2] A.-L. Barabási and R. Albert. Emergence of scaling in random networks. *Science*, 286(5439): 509–512, 1999. doi: 10.1126/science.286.5439.509.
- [3] L. Beghin and M. D’Ovidio. Fractional Poisson process with random drift. *Electron. J. Probab.*, 19:no. 122, 26, 2014. doi: 10.1214/EJP.v19-3258.
- [4] L. Beghin and E. Orsingher. Fractional Poisson processes and related planar random motions. *Electron. J. Probab.*, 14:no. 61, 1790–1827, 2009. doi: 10.1214/EJP.v14-675.
- [5] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996. ISBN 0-521-56243-0.
- [6] D. O. Cahoy and F. Polito. On a fractional binomial process. *J. Stat. Phys.*, 146(3):646–662, 2012. doi: 10.1007/s10955-011-0408-3.
- [7] D. O. Cahoy and F. Polito. Simulation and estimation for the fractional Yule process. *Methodol. Comput. Appl. Probab.*, 14(2):383–403, 2012. doi: 10.1007/s11009-010-9207-6.
- [8] D. O. Cahoy and F. Polito. Renewal processes based on generalized Mittag-Leffler waiting times. *Commun. Nonlinear Sci. Numer. Simul.*, 18(3):639–650, 2013. doi: 10.1016/j.cnsns.2012.08.013.
- [9] D. O. Cahoy and F. Polito. Parameter estimation for fractional birth and fractional death processes. *Stat. Comput.*, 24(2):211–222, 2014. doi: 10.1007/s11222-012-9365-1.
- [10] K. S. Crump. On point processes having an order statistic structure. *Sankhyā Ser. A*, 37(3): 396–404, 1975.
- [11] K. S. Crump. On point processes having an order statistic structure. *Sankhyā Ser. A*, 37(3): 396–404, 1975.
- [12] K. Diethelm. *The analysis of fractional differential equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. ISBN 978-3-642-14573-5. doi: 10.1007/978-3-642-14574-2. An application-oriented exposition using differential operators of Caputo type.
- [13] P. D. Feigin. On the characterization of point processes with the order statistic property. *J. Appl. Probab.*, 16(2):297–304, 1979.
- [14] P. D. Feigin and B. Reiser. On asymptotic ancillarity and inference for Yule and regular nonergodic processes. *Biometrika*, 66(2):279–283, 1979. doi: 10.1093/biomet/66.2.279.
- [15] R. Garra, R. Gorenflo, F. Polito, and Ž. Tomovski. Hilfer-Prabhakar derivatives and some applications. *Appl. Math. Comput.*, 242:576–589, 2014. doi: 10.1016/j.amc.2014.05.129.
- [16] R. Garra, E. Orsingher, and F. Polito. State-dependent fractional point processes. *J. Appl. Probab.*, 52(1):18–36, 2015. doi: 10.1239/jap/1429282604.
- [17] R. Gorenflo, A. A. Kilbas, F. Mainardi, and S. V. Rogosin. *Mittag-Leffler functions, related topics and applications*. Springer Monographs in Mathematics. Springer, Heidelberg, 2014. ISBN 978-3-662-43929-6; 978-3-662-43930-2. doi: 10.1007/978-3-662-43930-2.
- [18] N. L. Johnson, A. W. Kemp, and S. Kotz. *Univariate discrete distributions*. Wiley Series in Probability and Statistics. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, third edition, 2005. ISBN 978-0-471-27246-5; 0-471-27246-9. doi: 10.1002/0471715816.

- [19] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006. ISBN 978-0-444-51832-3; 0-444-51832-0.
- [20] A. E. Kyprianou. *Fluctuations of Lévy processes with applications*. Universitext. Springer, Heidelberg, second edition, 2014. ISBN 978-3-642-37631-3; 978-3-642-37632-0. doi: 10.1007/978-3-642-37632-0. Introductory lectures.
- [21] P. Lansky, F. Polito, and L. Sacerdote. The role of detachment of in-links in scale-free networks. *Journal of Physics A: Mathematical and Theoretical*, 47(34):345002, 2014.
- [22] N. Laskin. Fractional Poisson process. *Commun. Nonlinear Sci. Numer. Simul.*, 8(3-4): 201–213, 2003. doi: 10.1016/S1007-5704(03)00037-6. Chaotic transport and complexity in classical and quantum dynamics.
- [23] F. Mainardi, R. Gorenflo, and E. Scalas. A fractional generalization of the Poisson processes. *Vietnam J. Math.*, 32(Special Issue):53–64, 2004.
- [24] Y. E. Maruvka, N. M. Shnerb, D. A. Kessler, and R. E. Ricklefs. Model for macroevolutionary dynamics. *Proceedings of the National Academy of Sciences*, 110(27):E2460–E2469, 2013.
- [25] M. M. Meerschaert, E. Nane, and P. Vellaisamy. The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.*, 16:no. 59, 1600–1620, 2011. doi: 10.1214/EJP.v16-920.
- [26] M. F. Neuts and S. I. Resnick. On the times of births in a linear birthprocess. *J. Austral. Math. Soc.*, 12:473–475, 1971.
- [27] M. F. Neuts and S. I. Resnick. On the times of births in a linear birthprocess. *J. Austral. Math. Soc.*, 12:473–475, 1971.
- [28] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark, editors. *NIST handbook of mathematical functions*. U.S. Department of Commerce, National Institute of Standards and Technology, Washington, DC; Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-14063-8. With 1 CD-ROM (Windows, Macintosh and UNIX).
- [29] E. Orsingher and F. Polito. Fractional pure birth processes. *Bernoulli*, 16(3):858–881, 2010. doi: 10.3150/09-BEJ235.
- [30] E. Orsingher and F. Polito. The space-fractional Poisson process. *Statist. Probab. Lett.*, 82(4): 852–858, 2012. doi: 10.1016/j.spl.2011.12.018.
- [31] E. Orsingher and F. Polito. Randomly stopped nonlinear fractional birth processes. *Stoch. Anal. Appl.*, 31(2):262–292, 2013. doi: 10.1080/07362994.2013.759495.
- [32] A. Pachon, F. Polito, and L. Sacerdote. Random Graphs Associated to Some Discrete and Continuous Time Preferential Attachment Models. *J. Stat. Phys.*, 162(6):1608–1638, 2016. doi: 10.1007/s10955-016-1462-7.
- [33] M. Politi, T. Kaizoji, and E. Scalas. Full characterization of the fractional Poisson process. *EPL (Europhysics Letters)*, 96(2):20004, 2011.
- [34] P. S. Puri. On the characterization of point processes with the order statistic property without the moment condition. *J. Appl. Probab.*, 19(1):39–51, 1982.
- [35] P. S. Puri. On the characterization of point processes with the order statistic property without the moment condition. *J. Appl. Probab.*, 19(1):39–51, 1982.
- [36] W. J. Reed and B. D. Hughes. On the size distribution of live genera. *Journal of theoretical biology*, 217(1):125–135, 2002.

- [37] M. V. Simkin and V. P. Roychowdhury. Re-inventing Willis. *Phys. Rep.*, 502(1):1–35, 2011. doi: 10.1016/j.physrep.2010.12.004.
- [38] H. A. Simon. On a class of skew distribution functions. *Biometrika*, 42:425–440, 1955.
- [39] V. V. Uchaikin, D. O. Cahoy, and R. T. Sibatov. Fractional processes: from Poisson to branching one. *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, 18(9):2717–2725, 2008. doi: 10.1142/S0218127408021932.
- [40] G. U. Yule. A mathematical theory of evolution, based on the conclusions of dr. jc willis, frs. *Philosophical Transactions of the Royal Society of London. Series B, Containing Papers of a Biological Character*, pages 21–87, 1925.